J-REGULAR RINGS WITH INJECTIVITIES

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Abstract. A ring R is called a J-regular ring if $R/J(R)$ is von Neumann regular, where $J(R)$ is the Jacobson radical of R. It is proved that if R is J-regular, then (i) R is right n-injective if and only if every homomorphism from an n-generated small right ideal of R to R_R can be extended to one from R_R to R_R; (ii) R is right FP-injective if and only if R is right (J, R)-FP-injective. Some known results are improved.

1. **Introduction**

Throughout this paper rings are associative with identity. A ring R is *regular* means it is a von Neumann regular ring. We write J and S_r for the Jacobson radical $J(R)$ and the right socle of R, respectively. Let U be a set and $n \geq 1$, U_n denotes the set of all $n \times 1$ matrices with entries in U. A right ideal L of R is called *small* if, for any proper right ideal K of R, $L + K \neq R$.

Recall that a ring R is *right n-injective* if every homomorphism from an n-generated right ideal of R to R_R can be extended to one from R_R to R_R. R is *right F-injective* if R is right n-injective for every $n \geq 1$. And R is *right FP-injective* if every homomorphism from a finitely generated submodule of a free right R-module F_R to R_R can be extended to one from F_R to R_R. The left side of the above injectivities can be defined similarly. By restricting the ideals to small ones, in [**6**], the above injectivities are studied under the condition that R is a semiperfect ring with an essential right socle. In [**5**], the condition is weakened to that R is a semiregular ring. In this short article, the above two conditions are generalized to the one that R is a J-regular ring. Better results are obtained.

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2. **Results**

Definition 1. A ring R is *J-regular* if R/J is regular. It is obvious that regular rings are J-regular. But the converse is not true. For example, let $R = \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$ be the ring of upper triangular real matrices with all diagonal entries rational. Then $J(R) = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$. It is easy to see that R is J-regular but not regular.

Remark 2. Recall that a ring R is *semilocal* if R/J is a semisimple ring. R is *semiperfect* in case R is semilocal and idempotents lift modulo J. R is *semiregular* when R is J-regular and idempotents lift modulo J. So we have the following relations:

semiperfect $\Rightarrow$semilocal$\Rightarrow J$-regular,
semiperfect $\Rightarrow$semiregular$\Rightarrow J$-regular.

It is easy to show that J-regular rings are real generalizations of the above classes of rings. For example, let R_1 be a semilocal ring which is not semiregular and R_2 a semiregular ring that is not semilocal. Set $R = R_1 \prod R_2$. Since R_1 and R_2 are both J-regular, R is J-regular by the following Proposition 5. But R is neither semilocal nor semiregular.

A right ideal I of a ring R has a *weak supplement* in R if there exists a right ideal K of R such that $I + K = R$ and $I \cap K$ is a small right ideal of R.

Proposition 3. *The following are equivalent for a ring R:*
(1) *R is J-regular.*
(2) *Every principal right (or left) ideal of R has a weak supplement in R.*
(3) *Every finitely generated right (or left) ideal of R has a weak supplement in R.*

Proof. (1)$\Leftrightarrow$(2) is obtained by [**3**, Proposition 3.18]. It is obvious that (3)$\Rightarrow$(2). For (1)$\Rightarrow$(3), suppose that I is a finitely generated right ideal of R. Set $\overline{R} = R/J$. Since R is J-regular, $\overline{I}$ is a direct summand of $\overline{R}$. Then it is easy to get there is a right ideal K of R such that $I + K = R$ and $I \cap K \subseteq J$. Therefore, K is a weak supplement of I in R. $\square$

Proposition 4. *If R is J-regular, then every factor ring S of R is also J-regular.*

Proof. Let S be a factor ring of R and ϕ be the ring epimorphism from R to S. By [**1**, Corollary 15.8], $\phi(J) \subseteq J(S)$. So $S/J(S)$ is a factor ring of R/J. Since R/J is regular, $S/J(S)$ is regular. Thus, S is J-regular. $\square$

Proposition 5. *A direct product of rings $R = \prod_{i\in I} R_i$ is J-regular if and only if R_i is J-regular for every $i \in I$.*

Proof. By [**2**, Lemma 4.1], it is easy to see that $J=\prod_{i\in I} J_i$ where $J = J(R)$ and $J_i = J(R_i)$, $i \in I$. Since $\frac{R}{J} = \frac{\prod_{i\in I} R_i}{\prod_{i\in I} J_i} \cong \prod_{i\in I} \frac{R_i}{J_i}$, R/J is regular if and only if R_i/J_i is regular for every $i \in I$. So R is J-regular if and only if R_i is J-regular for every $i \in I$. $\square$

The following two propositions show that being J-regular is a Morita invariant.

Proposition 6. *If R is J-regular, then eRe is also J-regular, where $e^2 = e \in R$.*

Proof. We only need to show that for each $a \in eRe$, there exist $b \in eRe$ and $c \in J(eRe) = eJe$ (see [**2**, Theorem 21.10]) such that $a = aba + c$. As R is J-regular, there exist $b' \in R$ and $c' \in J$ such that $a = ab'a + c'$. Since $a \in eRe$, $a = ab'a + c' = aeb'ea + c'$. It is clear that $c' = a - ab'a \in eRe \cap J = eJe$. Then we can set $b = eb'e$ and $c = c'$. $\square$

Proposition 7. *If R is J-regular, then every matrix ring $\mathrm{M}_{n\times n}(R)$ is also J-regular, $n \geq 1$.*

Proof. It is well-known that $J(\mathrm{M}_{n\times n}(R)) = \mathrm{M}_{n\times n}(J)$ (see [**2**, Page 61]). And it is also easy to prove that $\frac{\mathrm{M}_{n\times n}(R)}{\mathrm{M}_{n\times n}(J)} \cong \mathrm{M}_{n\times n}(\frac{R}{J})$. Therefore $\frac{\mathrm{M}_{n\times n}(R)}{J(\mathrm{M}_{n\times n}(R))} = \frac{\mathrm{M}_{n\times n}(R)}{\mathrm{M}_{n\times n}(J)} \cong \mathrm{M}_{n\times n}(\frac{R}{J})$. Since R is J-regular, R/J is a regular ring. So $\mathrm{M}_{n\times n}(\frac{R}{J})$ is also regular. Thus, $\mathrm{M}_{n\times n}(R)$ is J-regular. $\square$

Theorem 8. *Let R be a J-regular ring and K a finitely generated projective right R-module. Then the endomorphism ring* End (K) *of K is also J-regular.*

Proof. Since K is finitely generated and projective, K is a direct summand of a finitely generated free right R-module F. Then there exists some integer $n \geq 1$

such that $\mathrm{End}(F)\cong \mathrm{M}_{n\times n}(R)$ and $\mathrm{End}(K)\cong e\mathrm{M}_{n\times n}(R)e$ for some idempotent e in $\mathrm{M}_{n\times n}(R)$. Thus, by Proposition 6 and Proposition 7, $\mathrm{End}(K)$ is J-regular. □

Now we turn to the main results. The following lemma is inspired by [**3**, Lemma 3.4].

Lemma 9. *Let R be a ring, $b, r_i, a_i \in R$, $i = 1, 2, \ldots, n$, such that $b + \sum_{i=1}^n a_i r_i = 1$. Then $bR \cap \sum_{i=1}^n a_i R = \sum_{i=1}^n ba_i R$.*

Proof. Assume that $x \in bR \cap \sum_{i=1}^n a_i R$. And set $c = \sum_{i=1}^n a_i r_i$. Then there exist $t, t_1, \ldots, t_n \in R$ such that $x = bt = (1-c)t = \sum_{i=1}^n a_i t_i$. Thus $t = ct + \sum_{i=1}^n a_i t_i \in \sum_{i=1}^n a_i R$. So $x = bt \in \sum_{i=1}^n ba_i R$. Conversely, $\sum_{i=1}^n ba_i R = \sum_{i=1}^n (1-c) a_i R \in bR \cap \sum_{i=1}^n a_i R$. □

Corollary 10. ([**3**, Lemma 3.4]) *Let R be a ring, $r, a \in R$ and $b = 1 - ar$. Then $bR \cap aR = baR$.*

Theorem 11. *If R is J-regular and $n \geq 1$, then R is right n-injective if and only if every homomorphism from an n-generated small right ideal of R to R_R can be extended to one from R_R to R_R.*

Proof. The necessity is obvious. For the sufficient part, assume that $I = a_1 R + \cdots + a_n R$ is an n-generated right ideal of R and f is a homomorphism from I to R_R. Since R is J-regular, by Proposition 3, I has a weak supplement in R. Thus, there exists a right ideal K of R such that $I + K = R$ and $I \cap K \subseteq J$. It is easy to see there are $r_1, \ldots, r_n \in R$, $b \in K$ such that $b + \sum_{i=1}^n a_i r_i = 1$ and $I \cap bR \subseteq I \cap K \subseteq J$. Therefore, $I \cap bR$ is a small right ideal of R. By Lemma 9, $I \cap bR = \sum_{i=1}^n ba_i R$ is n-generated. Thus, by hypothesis, there is a homomorphism g from R_R to R_R such that $g_{|I\cap bR} = f_{|I\cap bR}$. Since $I + bR = R$, for each $x \in R$, there exist $x_1 \in I$, $x_2 \in bR$ such that $x = x_1 + x_2$. Define a map F from R_R to R_R with $F(x) = f(x_1) + g(x_2)$ for each $x \in R$. It is easy to prove that F is a well-defined homomorphism from R_R to R_R such that $F_{|I} = f$. □

Corollary 12. *If R is J-regular, then R is right F-injective if and only if every homomorphism from a finitely generated small right ideal of R to R_R can be extended to one from R_R to R_R.*

Let I, K be two right ideals of a ring R and $m \geq 1$. In [**6**], R is called a *right* (I, K)*-m-injective* ring if, for any m-generated right ideal $U \subseteq I$, every homomorphism from U to K can be extended to one from R_R to R_R. And R is *right* $(I,\ K)$*-FP-injective* if, for any $n \geq 1$ and any finitely generated right R-submodule N of I_n which is a submodule of the free right R-module R_n, every homomorphism from N to K can be extended to one from R_n to R_R.

Using the same method in the proof of Theorem 11, we have the following result.

Theorem 13. *Let K be a right ideal of a J-regular ring R and $m \geq 1$. Then R is right (R, K)-m-injective if and only if R is right (J, K)-m-injective.*

Lemma 14. ([**6**, Lemma 1.3]) *The following are equivalent for two right ideals I and K of a ring R:*
(1) *R is right $(I,\ K)$-FP-injective.*
(2) $\mathrm{M}_{n\times n}(R)$ *is right* $(\mathrm{M}_{n\times n}(I),\ \mathrm{M}_{n\times n}(K))$*-1-injective for every $n \geq 1$.*

Theorem 15. *If R is J-regular, then R is right FP-injective if and only if R is right $(J,\ R)$-FP-injective.*

Proof. If R is right FP-injective, it is clear that R is right (J, R)-FP-injective. Conversely, assume that R is right (J, R)-FP-injective. By Lemma 14, $\mathrm{M}_{n\times n}(R)$ is right $(\mathrm{M}_{n\times n}(J),\ \mathrm{M}_{n\times n}(R))$-1-injective for every $n \geq 1$. Since R is J-regular, by Proposition 7, $\mathrm{M}_{n\times n}(R)$ is J-regular. Again since $J(\mathrm{M}_{n\times n}(R))=\mathrm{M}_{n\times n}(J)$, Theorem 11 implies that $\mathrm{M}_{n\times n}(R)$ is right 1-injective for every $n \geq 1$. Thus, by [**4**, Theorem 5.41], R is right FP-injective. □

By the above theorems, we obtain the following corollaries.

Corollary 16. *Let R be a semilocal ring.*
(1) *If I is a right ideal of R and $m \geq 1$, then R is right (R, I)-m-injective if and only if R is right (J, I)-m-injective.*
(2) *R is right F-injective if and only if R is right (J, R)-n-injective for every $n \geq 1$.*
(3) *R is right FP-injective if and only if R is right (J, R)-FP-injective.*

Remark 17. Recall that a ring R is *right small injective* if every homomorphism from a small right ideal of R to R_R can be extended to one from R_R to

R_R. It was proved in [**5**, Theorem 3.16 (1)] that if R is semilocal, then R is right self-injective if and only if R is right small injective. But the results in the above corollary weren't obtained in [**5**].

Corollary 18. ([**5**, Theorem 3.16 (3), (4)]) *Let R be a semiregular ring and $m \geq 1$.*
(1) *If I is a right ideal of R, then R is right (J, I)-m-injective if and only if R is right (R, I)-m-injective.*
(2) *R is right (J, R)-FP-injective if and only if R is right FP-injective.*

Corollary 19. ([**6**, Lemma 2.3]) *Let R be a semiregular ring.*
(1) *If R is right (J, S_r)-1-injective, then R is right (R, S_r)-1-injective.*
(2) *If R is right (J, R)-1-injective, then R is right 1-injective.*

Corollary 20. ([**6**, Theorem 2.11 (1), (2)]) *Let R be a semiperfect ring with an essential right socle and $m \geq 1$.*
(1) *If R is right (J, S_r)-m+1-injective, then R is right (R, S_r)-m-injective.*
(2) *If R is right (J, R)-m+1-injective, then R is right m-injective.*

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